Notes 2: Probability

Julio Garín
Department of Economics

Statistics for Economics

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Some Concepts

The *sample space* for an experiment is set of all possible experimental outcomes.

Table: Example

Experiment	**Sample Space** (S)
Toss a coin	Head, Tail

- Graphical Representation.

Counting: Combinations

Principles

Multiplication Principle: If an experiment can be described as a sequence of k steps with n_1 possible outcomes on the first step, n_2 possible outcomes in the second step, and so on, then the total number of experimental outcomes is given by $n_1 \times n_2 \times \cdots n_k$

Addition Principle: If two events E_1 and E_2 can occur independently in exactly m ways and n ways respectively, then either of the two events can occur in $(m + n)$ ways.

Introduction to Counting

1. Given the letters A, B, C, D and E.
   - How many strings of length 4 can be formed if repetitions are not allowed?
   - How many of those begin with the letter C?
2. A sequence of 5 cards is drawn from a deck. After each card is drawn, it is then replaced.
   - How many sequences are possible?
   - How many sequences do not contain any kings and queens?
   - How many sequences contain at least one king and one queen?

Either Salad or Change

- Objective: count objects in no particular order.

$$C_r^n \equiv {}_nC_r \equiv \binom{n}{r} \equiv C(n,r) = \frac{n!}{r!(n-r)!}$$

- Two types of combinations:
  - With replacement.
  - ... replacement.
- Example: Powerball (5 out of 59 + 1 out of 39).

Combinations Without Replacement: Example

- A poker hand consists of 5 cards dealt from an ordinary deck of 52 playing cards.
  1. How many poker hands are possible?
  2. A full house consists of 3 cards of one denomination and 2 cards of another. How many different full houses are possible?
  3. Calculate the probability of being dealt a full house.
- There are five women and six men in a group. From this group a committee of 4 is to be chosen. How many different ways can a committee be formed that contain at least three women?

Counting: Permutations

School, Not Salad Anymore

- Now order matters.

$$P_r^n \equiv {}_nP_r \equiv r!\binom{n}{r} \equiv P(n,r) = \frac{n!}{(n-r)!}$$

- Proof?
- Two types of Permutations:
    - With replacement (n^r).
    - ... replacement (!)

Combinations Without Replacement: Problems

1. Pick 3 songs from a 9-song CD and play them in some order.
2. How many different signals can be made by 5 flags from 8-flags of different colours?
3. In an exacta wager at the race track, a bettor picks the two horses that she thinks will finish first and second in a specified order. For a race with 12 entrants, determine the number of exacta wagers.

Probability.

Concepts

An *experiment* is the process by which an observation is made.

A *simple event* is an event that cannot be decomposed. Each simple event corresponds to one and only one *sample point*. E_i denotes a simple event.

The *sample space*, S, associated with an experiment is the set consisting of all possible sample points.

Suppose S is a sample space associated with an experiment. To every event A in S, we assign a number, $P(A)$, called the probability of A, so the following axioms hold:

1. $P(A) \geq 0$
2. $P(S) = 1$
3. $A_1, A_2, \cdots, A_n$ form a sequence of pairwise mutually exclusive events in S, then,

$$P\left(A_i \cup A_2 \cup \cdots \cup A_n\right) = \sum_{i=1}^{n} P(A_i)$$

Approaches

- Subjective.
- Frequentist.
  - Distribution of outcomes in a finite sample.
- **Classical**.
  - $\frac{f}{N}$ 'rule'.

What generates probabilities?

- Repeated observation.
  - Sex of babies (4 million birth per year).
    - $\Pr(\text{Boy}) = 0.52$
    - $\Pr(\text{Girl}) = 0.48$
- Theory:
  - $\Pr(\text{Head}) = 0.5$
  - $\Pr(1 \text{ on a dice}) = \frac{1}{6}$

Example of Limiting Relative Frequency

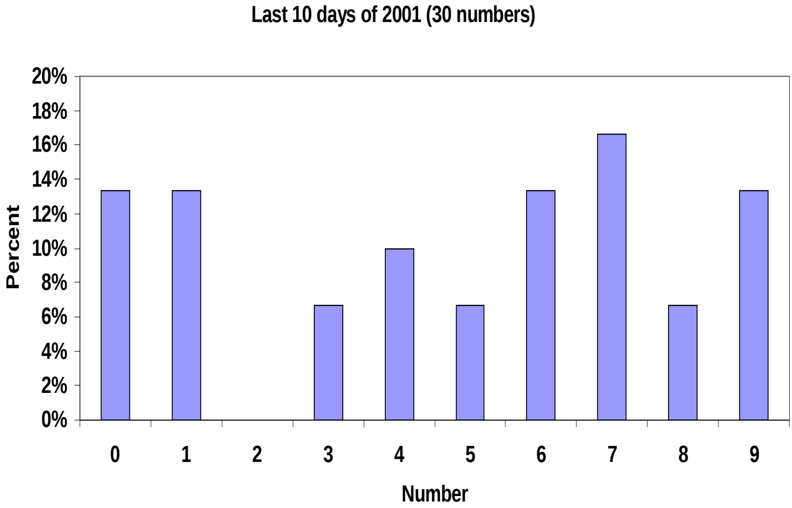
- Maryland Lottery. Daily number: Pick 3 numbers between 0-0-0 and 9-9-9.
  - $1 to play.
  - If you match the number exactly, you win $500.
- Probability: how often would we expect integers $(0, 1, 2, \cdots)$ to show up?
- Frequency: how often do they actually show up?
- If the game is fair, what should happen as the number of draws increase?

Example: Maryland Lottery

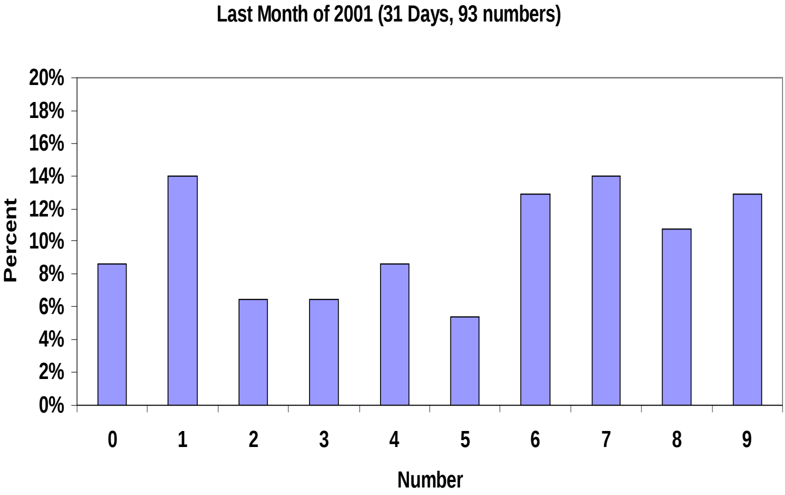
Lottery: A tax on people who are bad at math.

- Started July 29, 1976.
- First game: Pick 3 evening number.
- Look at the frequency of numbers drawn for Pick 3 evening number.
- What should we expect?

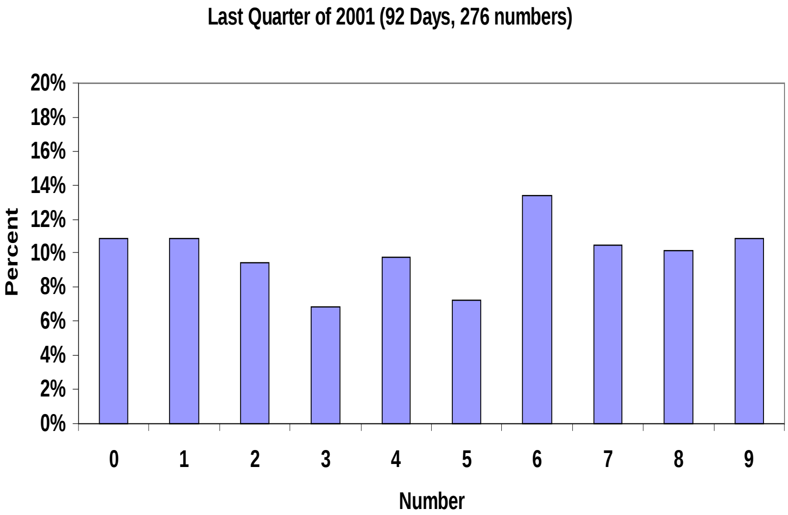
Example: Maryland Lottery



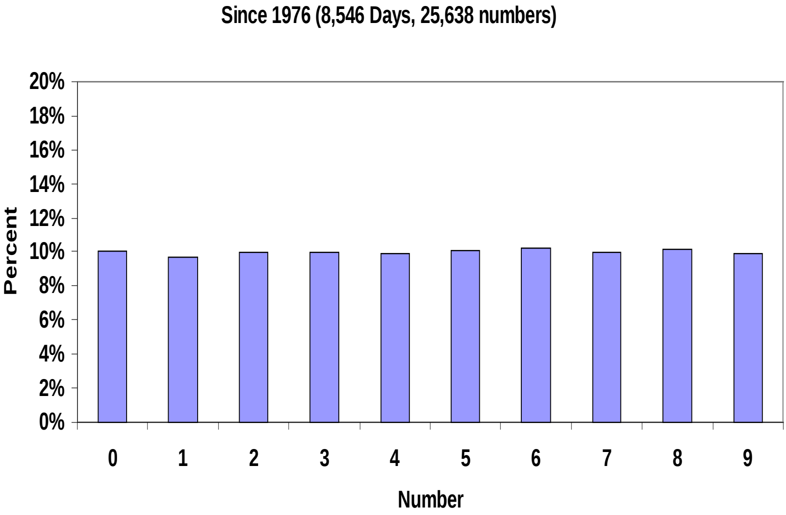
Example: Maryland Lottery



Example: Maryland Lottery



Example: Maryland Lottery



Basics of Set Theory and Some Concepts

- Review of Set Theory.
  - Compound events: union and intersection.
  - Complement.

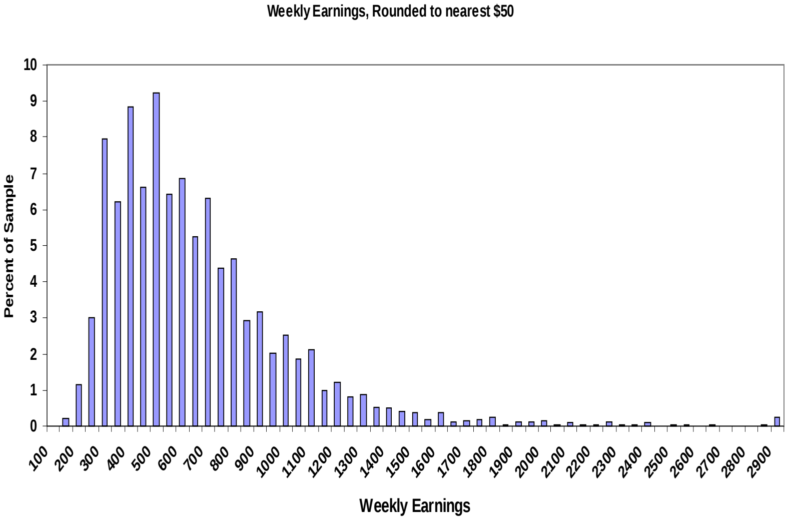
$$P(A) + P(A^C) = 1$$

Two events are said to be *mutually exclusive* if the events have no sample points in common.

- Special Addition Rule.

$$P(A \cup B) = P(A) + P(B)$$

Mutually Exclusive Events and the Relative Frequency



Non Mutually Exclusive or Compatible Events

- Intuition.
- General addition rule.

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

- Example: $P(\text{Spade or Face})$

Example I: The Famous Birthday Problem

What is the chance that at least 2 people in a group of k have the same birthday?

- What are the possible outcomes?
- Rephrasing: What are the number of possible combinations where no two birthdays are the same?
  - Use complements!

Table: Solution

k	$P(1 \text{ or more matches})$
15	0.253
40	0.891
50	0.970
70	0.999

Example II: 2000 Census

- Federal Surveys ask respondents questions about:
  - Race.
    - White, African American, Asian, American Indian/Eskimo, Pac. Isl.
    - Allows for multiple codes.
  - Ethnicity.
    - Hispanic or not.
- From the 2000 Census:
  - 12.5% of the population is Hispanic.
  - 75.1% of the population reports White only as a race.
  - 6.0% of the population reports White and Hispanic.
- Pr(W or H)?
- Minority is defined as Hispanic or Non-white. Pr(Minority)?

Conditional Probability

Table: Promotion Status of Policy Officers

	Men	Women	Total
Promoted	288	36	324
Not Promoted	672	204	876
Total	960	240	1200

- There was discrimination?
- Joint probabilities.
- Marginal probabilities.

The *conditional probability of an event* A, given that event B has occurred, is equal to

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

provided $P(B) > 0$.

Example I: Poverty and Education

Table: Poverty and Education

Poverty	Less than High School	More than High School	Totals
No	0.124	0.785	0.908
Yes	0.035	0.056	0.092
Total	0.159	0.841	1.000

1. Pr(Poverty)?
2. $\Pr(\text{Poverty}| < HS)$?
3. $\Pr(\text{Poverty}| > HS)$?
4. Conclusions?

Example II: Benefit of Seat Belts

DOT HS 809 090

U.S. Department of Transportation
National Highway Traffic
Safety Administration

Traffic Safety Facts 1999

Occupant Protection

"Safety belts, when used, reduce the risk of fatal injury to front-seat passenger car occupants by 45 percent."

Example II: Benefit of Seat Belts

- Data Fatal Accident Reporting System.
  - Census of all motor vehicle accidents that generate a fatality.
  - Collect data on vehicle occupants:
    - Did they die in the crash?
    - Seating position?
    - Age and sex.
    - Wearing a seat belt.
- Our concern is the effectiveness of belts in crashes that can kill.
- Need a sample of violent crashes.
- What is the benchmark?
- 17,375 Crashes from 1977 to 1984.

Example II: Benefit of Seat Belts

Table: Belt use and Driver Fatality

Driver Died	$\neg$ **Use Belt**	**Use Belt**	**Totals**
No	0.6600	0.0280	0.6880
Yes	0.3060	0.0059	0.3119
Total	0.9661	0.0339	1.0000

1. $\Pr(\text{Died}|\neg\text{Belt})$?
2. $\Pr(\text{Died}|\text{Belt})$?
3. What is the main question we are trying to address?
   - What is the reduction in the probability of death in a serious crash?

Independent Events

- Motivation.
- Formal definition.
- **Multiplicative law** for dependent and independent events.
- **Additive law** for dependent and independent events.

Example I: Coffee

- Three brands of coffee, X, Y, and Z, are to be ranked according to taste by a judge. Define the following events:
  1. Brand X is preferred to Y.
  2. Brand X is ranked best.
  3. Brand X is ranked second best.
  4. Brand X is ranked third best.

  If the judge actually has no taste preference and randomly assigns ranks to the brands, is event A independent of events B, C, and D?

Example II: Should I Study for the Midterm?

Table: Scores on Exams

Final	<A	A	Totals
<A	0.647	0.059	0.706
A	0.059	0.235	0.294
Total	0.706	0.294	1.000

1. $\Pr(\text{A on Final}|\text{A on Midterm})$?
2. $\Pr(\text{A on Final})$?
3. Conclusions?

Example III: Does the Sex of Your First Child Tell you Anything About the sex of the Second?

Table: Sex of the First and Second Child

Second	**Girl**	**Boy**	Totals
Girl	0.2397	0.2492	0.4889
Boy	0.2467	0.2653	0.5111
Total	0.4864	0.5163	1.000

1. $\Pr(\text{Boy on second} | \text{Girl on first})$?
2. $\Pr(\text{Boy on second})$?
3. $\Pr(\text{Girl on second} | \text{Girl on first})$?
4. $\Pr(\text{Girl on second})$?

Example IIIa: Is Fertility Related to the Sex of the First Child?

- Sample of women with at least 1 child.

Table: Have a Second Kid?

Sex of the first	**No**	**Yes**	**Totals**
Girl	0.149	0.339	0.488
Boy	0.157	0.355	0.512
Total	0.306	0.694	1.000

1. Pr(Second Child|First is a Boy)?
2. Pr(Second Child)?
3. Conclusions?

Example IIIb: Is Fertility Related to the Sex Mix of the First Two Kids?

Table: Have a Third Kid?

Sex of the first two kids	No	Yes	Totals
Boy & Girl	0.310	0.184	0.494
2 Boys or 2 Girls	0.287	0.219	0.506
Total	0.597	0.403	1.000

1. Pr(Third Kid|Boy and Girl)?
2. Pr(Third Kid|2 Boys or 2 Girls)?
3. Pr(Third Child)?
4. Conclusions?

Dependence and Small Probabilities

- Do not confuse them!
- *Do not* confuse them!
- **Do not** confuse them!
- Example: *The World According to Garp*:
  - A = First plane crashes into a house.
  - B = Second plane crashes into the same house.
  - $\Pr(B) = 0.00001$
  - $\Pr(A \cap B)$ is very small.
  - Why has $\Pr(B)$ fallen?

Dependence and Small Probabilities

- Brian and Charlie live in the same neighborhood and both play the lottery.
- Brian wins the lottery.
- Charlie quits playing lottery saying "When have you ever heard of two people winning from the same neighborhood".
- What is wrong with Charlie?

Bayes' Theorem

- Who cares?

Law of Total Probability

For some positive integer k, let the sets $B_1, B_2, \cdots, B_k$ be such that,

1. $S = B_1 \cup B_2 \cup \cdots B_k$.
2. $B_i \cap B_j = 0$, for $i \neq j$.

Then the collection of sets $\{B_1, B_2, \cdots, B_k\}$ is said to be a **partition** of S.

If A is any subset of S, and $\{B_1, B_2, \cdots, B_k\}$ is a partition of S, A can be decomposed as follows,

$$A = (A \cap B_1) \cup (A \cap B_2) \cup \cdots \cup (A \cap B_k)$$

Theorem LofTP: Assume that $\{B_1, B_2, \cdots, B_k\}$ is a partition of S, such that $P(B_i) > 0$, for $i = 1, 2, \cdots, k$. Then, for any event A

$$P(A) = \sum_{i=1}^{k} P(A|B_i)\, P(B_i)$$

Example

In box 1, there are 60 short bolts and 40 long bolts. In box 2, there are 10 short bolts and 20 long bolts. Take a box at random, and pick a bolt.

- What is the probability that you chose a short bolt?

Let B_i = choose Box 1.
Therefore,

$$P\left(\text{Short}\right) = P\left(\text{Short}\middle|B_1\right)P(B_1) + P\left(\text{Short}\middle|B_2\right)P(B_2) = \frac{60}{100}\frac{1}{2} + \frac{10}{30}\frac{1}{2}$$

Bayes' Rule

Bayes' Rule: Assume that $\{B_1, B_2, \cdots, B_k\}$ is a partition of S, such that $P(B_i) > 0$, for $i = 1, 2, \cdots, k$. Then,

$$P(B_j|A) = \frac{P(A|B_j)\,P(B_j)}{\sum_{i=1}^{k} P(A|B_i)\,P(B_i)}$$

Example I: Small Cards

Small cars may not be as safe as large cars. Small cars accounted for 18% of all vehicles on the road. Accidents involving small cars led to 11,898 fatalities during a recent year. Assume that the probability that a small car is involved in an accident is 0.18 and that the probability that an accident involving a small car leads to a fatality is 0.128. Similarly, the probability that the accident leads to a fatality driving a car that is not small is 5%. You heard of an accident involving a fatality. Assume that the probability of getting into an accident is independent of car size. What is the probability that a small car was involved?

1. What is the probability that a small car was involved?

Example II: False Positive

Two different tests, A and B are available for detecting the presence of HIV virus in human blood. Test A is more sensitive than Test B: when the HIV virus is present (event $H+$), Test A gives a positive result (event $T+$) with probability 0.999 whereas Test B gives a positive result with probability only 0.99.
On the other hand, Test B is more specific than Test A: when the HIV virus is not present (event H), Test B gives positive result $T+$ with probability only 0.001 whereas Test A gives positive result $T+$ with larger probability, 0.01. For obvious reasons, such positive results are called *false positives*.
False positive results are a huge nuisance in medical testing (and testing in general) because they can lead to the patient receiving unnecessary treatment. In HIV testing, false positive results can also lead to undesirable familial, social, and societal consequences. *False negative* results (event T occurring when the virus is actually present) also have undesirable consequences in that a patient may not get treatment for the disease, and may also unknowingly infect others.

1. Suppose that 2% of the population has HIV virus in their bloodstream, find the value of $P(T+)$ for each test.
2. For each test, compute $P(H+|T+)$.
3. For each test, compute $P(H+|T-)$.

Example III: No-Name Example and Tabular Approach

The prior probabilities for events A_1, A_2 and A_3 are $\Pr(A_1) = 0.2$, $\Pr(A_2) = 0.5$, and $\Pr(A_3) = 0.3$. The conditional probabilities of B given A_1, A_2 and A_3 are $\Pr(B|A_1) = 0.5$, $\Pr(B|A_2) = 0.4$, and $\Pr(B|A_3) = 0.3$.

1. Find $\Pr(B \cap A_1), \Pr(B \cap A_2)$ and $\Pr(B \cap A_3)$.
2. Apply Bayes' theorem to compute the posterior probability $\Pr(A_2|B)$.
3. Use the tabular approach to applying Bayes' theorem to compute $\Pr(A_1|B), \Pr(A_2|B)$, and $\Pr(A_3|B)$.

Example IV: Multiple Choice Test

A student answers a multiple-choice examination question that offers four possible answers. Suppose the probability that the student knows the answer to the question is 0.7 and the probability that the student will guess is 0.3. Assume that if the student guesses, the probability of selecting the correct answer is 0.35.

1. What is the unconditional probability that the student is correct?
2. If the student correctly answers a question, what is the probability that the student really knew the correct answer?

Example V: Monty Hall Problem

Example IV: Monty Hall Problem

Suppose you're on a game show, and you're given the choice of three doors: Behind one door is a car; behind the others, goats. You pick a door, say No. 1, and the host, who knows what's behind the doors, opens another door, say No. 3, which has a goat. He then says to you, "Do you want to pick door No. 2?" Is it to your advantage to switch your choice?
Whitaker & vos Savant (1990)

- What would you do?