Economics 30330: Statistics for Economics
Problem Set 4 - Suggested Answers
University of Notre Dame
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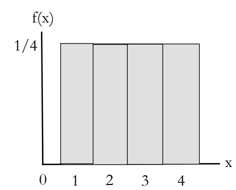
Discrete Probability Distributions (80 points)

1. A technician services computers for faculty members at the University of Notre Dame. Depending on the type of malfunction, the service call can take 1, 2, 3, or 4 hours. The different types of malfunctions occur at about the same frequency.

   (a) Develop a probability distribution for the duration of a service call and graph the distribution.

$$f(x) = \begin{cases} \frac{1}{4} & 0 \leq x \leq 4 \\ 0 & \text{elsewhere} \end{cases}$$



   (b) Show that your probability function satisfies conditions required for a discrete probability function (p.f.).

      i. $1/4 \geq 0 \Rightarrow f(x) \geq 0 \quad \forall x$

      ii. $\sum_{i=1}^{4} 1/4 = 1 \Rightarrow \sum_x f(x) = 1$

   (c) What is the probability that a service call will take 3 hours?
   $\Pr(X = 3) = f(x = 3) = f(3) = 1/4 = 0.25$.

   (d) A service call has just come in but the type of malfunction is unknown. It is 3:00 P.M. and technicians usually get done at 5:00 P.M. The technician always takes his dog out for a walk immediately after work, and his dog gets "upset" when he is late. What is the probability of an upset dog tonight?
   $\Pr(\textit{Upset dog}) = \Pr(X = 3 \cup X = 4) = f(3) + f(4) = 1/4 + 1/4$. *Therefore,* $\Pr(\textit{Upset dog}) = 0.5$.

2. Suppose the national unemployment rate is 4.1%. 100 members of the labor force are selected at random.

   (a) Of these 100 people, what is the expected number of unemployed?
   If we let success be the event of selecting a member of the labor force unemployed, this experiment can be described by a binomial distribution.
   We have then that $\mu = n \cdot p = (100)(0.041)$. *Therefore,* $\mathrm{E}(x) = 4.1$.

   (b) What is the standard deviation of the number of unemployed?

$$\begin{aligned} \mathrm{Var}(x) &= n \cdot p \cdot q \\ &= (100)(0.041)(0.959) \\ &= 3.93 \end{aligned}$$

   Therefore $\sigma = \sqrt{3.93} = 1.98$.

3. A 7-Eleven store serves 0 to 5 customers from 10 to 11:00A.M. The p.d.f. of the number of customers in that hours is:

Number of Customers	Probability	Number of Customers	Probability
0	0.1	3	0.20
1	0.15	4	0.15
2	0.30	5	0.10

(a) What is the expected number of customers between 10 and 11?

$$\begin{aligned} \text{E}(x) &= \sum_x x f(x) \\ &= (0)(0.1) + (1)(0.15) + (2)(0.3) + (3)(0.2) + (4)(0.15) + (5)(0.1) \\ &= 2.45 \end{aligned}$$

(b) What is the variance and the standard deviation of the number of customers?

$$\begin{aligned} \text{Var}(x) &= \sum_x (x-\mu)^2 f(x) \\ &= (0-2.45)^2(0.1) + (1-2.45)^2(0.15) + (2-2.45)^2(0.3) \\ &\quad + (3-2.45)^2(0.2) + (4-2.45)^2(0.15) + (5-2.45)^2(0.1) \\ &= 2.0475 \end{aligned}$$

Therefore $\sigma = \sqrt{2.0475} = 1.4309$

4. Nine percent of undergraduate college students have a balance on their credit card that exceeds \$7000. 10 undergraduates are selected at random and are interviewed about their credit card usage.

(a) Is the selection of these 10 students a binomial experiment? Explain.

Yes, because it satisfies the characteristics of that type of experiment. Specifically:

- *We have n identical trials (10).*
- *Each trial results in one of two outcomes (balance exceeds \$7,000 or not).*
- *Probability of success (balance exceeding \$7,000) is always the same (0.09).*
- *Independent trials (students are selected at random).*

Therefore, the P.D.F. is given by $f(x) = C_x^n p^x q^{n-x}$. Specifically in this case we have

$$f(x) = C_x^{10}(0.09)^x(0.91)^{10-x}$$

(b) What is the probability that 2 of these 10 students will have a credit card balance that exceeds \$7000 (each)?

$$\begin{aligned} \Pr(X=2) &= f(2) \\ &= C_2^{10}(0.09)^2(0.91)^8 \\ &= 0.1714 \end{aligned}$$

(c) What is the probability that none of them will have a credit card balance in excess of \$7000?

$$\begin{aligned} \Pr(X=0) &= f(0) \\ &= C_0^{10}(0.09)^0(0.91)^{10} \\ &= 0.3894 \end{aligned}$$

(d) What is the probability that at least 3 will have a credit card balance that exceeds \$7000?

$$\begin{aligned}
\Pr(X \geq 3) &= 1 - \Pr(X \leq 2) \\
&= 1 - F(2) \\
&= 1 - \sum_{i=0}^{2} f(x_i) \\
&= 1 - [f(0) + f(1) + f(2)] \\
&= 1 - (0.3894 + 0.3851 + 0.1714) \\
&= 1 - 0.9459 \\
&= 0.0541
\end{aligned}$$

5. In the 2002 baseball season, Barry Bonds had the highest batting average in the major leagues at 0.370. Suppose the probability Barry Bonds gets a hit during a particular at bat is 0.37. If Bonds gets 5 at bats in a game, what is the chance he:

(a) Will get exactly 3 hits?

This is a binomial experiment with $p = 0.37$ *and* $n = 5$. *Since for a binomial experiment the P.D.F. is given by* $f(x) = {}_nC_x p^x q^{n-x}$. *In this case we have*

$$f(x) = {}_5C_x (0.37)^x (0.63)^{5-x}$$

Hence,

$$\begin{aligned}
\Pr(X = 3) &= f(3) \\
&= {}_5C_3 (0.37)^3 (0.63)^2 \\
&= 0.201
\end{aligned}$$

(b) Will get at least 1 hit?

$$\begin{aligned}
\Pr(X \geq 1) &= 1 - \Pr(X < 1) \\
&= 1 - f(0) \\
&= 1 - {}_5C_0 (0.37)^0 (0.63)^5 \\
&= 1 - (0.63)^5 \\
&= 0.901
\end{aligned}$$

6. Airline passengers arrive randomly and independently at the passenger-screening facility at O'Hare International Airport. The mean arrival rate is 10 passengers per minute.

(a) How is this experiment distributed?

Let X *be the number of passengers arriving per minute. Hence,* $X \sim Pois(10, 10)$ *or*

$$f(x) = \frac{10^x}{x!} e^{-10}$$

(b) What is the probability of no arrivals in a one-minute period?

$$\begin{aligned}
\Pr(X = 0) &= f(0) \\
&= \frac{10^0}{0!} e^{-10} \\
&= 0.0000454
\end{aligned}$$

(c) What is the probability that fewer than 4 passengers arrive in a one-minute period?

$$\begin{aligned}\Pr(X < 4) &= 1 - \Pr(X \leq 3) \\ &= F(3) \\ &= \sum_{i=0}^{3} f(x_i) \\ &= \sum_{k=0}^{3} \frac{10^k}{k!} e^{-10} \\ &= 0.0103\end{aligned}$$

(d) What is the probability that no passengers arrive in a 15 second period?

Before we had 10 passengers per minute now, however, the new λ is given by

$$\lambda' = 10\left(\frac{15}{60}\right) = 2.5$$

Hence,

$$\begin{aligned}\Pr(X < 0) &= f(0) \\ &= F(3) \\ &= \frac{2.5^0}{0!} e^{-2.5} \\ &= 0.0821\end{aligned}$$

(e) What is the probability of at least one passenger arrives in a 15 second period?

$$\begin{aligned}\Pr(X \geq 1) &= 1 - \Pr(X \leq 0) \\ &= 1 - f(0) \\ &= 1 - 0.0821 \\ &= 0.9179\end{aligned}$$

7. In the game of Blackjack, a player is initially dealt 2 cards. Face cards (jacks, queens, kings) and 10s have a point value of 10. Aces have a point value of either 1 or 11. A 52-card deck contains 16 cards with a point value of 10 and 4 aces. In the following, assume that two cards are drawn at random from the deck.

This is a hypergeometric experiment with $N = 52$ and $n = 2$. Remember,

$$f(x) = \frac{\binom{r}{x}\binom{N-r}{n-x}}{\binom{N}{n}}$$

(a) What is the probability that both cards are aces or 10-point cards (or both)?

In this case we have $r = 20$. So that,

$$\begin{aligned}\Pr(X = 2) &= f(2) \\ &= \frac{\binom{20}{2}\binom{52-20}{2-2}}{\binom{52}{2}} \\ &= \frac{\binom{20}{2}\binom{30}{0}}{\binom{52}{2}} \\ &= 0.1433\end{aligned}$$

(b) What is the probability that both cards are aces?

Now $r = 4$. Hence,

$$\begin{aligned}\Pr(X = 2) &= f(2)\\ &= \frac{\binom{4}{2}\binom{52-4}{2-2}}{\binom{52}{2}}\\ &= \frac{\binom{4}{2}\binom{48}{0}}{\binom{52}{2}}\\ &= 0.0045\end{aligned}$$

(c) What is the probability that both cards are 10-point cards?

We have here that $r = 16$. Therefore,

$$\begin{aligned}\Pr(X = 2) &= f(2)\\ &= \frac{\binom{16}{2}\binom{52-16}{2-2}}{\binom{52}{2}}\\ &= \frac{\binom{16}{2}\binom{36}{0}}{\binom{52}{2}}\\ &= 0.0905\end{aligned}$$

(d) A Blackjack is a 10-point card and an ace for a value of 21. Use your answers to parts *a*) – *c*) above to find the probability that a player is dealt a Blackjack.

By definition, the probability of a Blackjack is given by the probability of having one ace and the probability of having a 10-point card. In order to obtain that intersection, we need to subtract the probability that both cards will happen. That is,

$$\begin{aligned}\Pr(\text{Blackjack}) &= \Pr(\text{10-point} \cap \text{Ace})\\ &= \Pr(\text{10-point} \cup \text{Ace}) - \Pr(\text{Both are 10-points}) - \Pr(\text{Both are aces})\\ &= 0.1433 - 0.0045 - 0.0905\\ &= 0.0483\end{aligned}$$

8. Given $Z = \alpha_0 + \alpha_1 W$, where Z and W are random variables. Derive:

For this exercise you should be familiar with the properties of moments that we covered in class.

(a) Expected value of Z.

$$\begin{aligned}\mathrm{E}(Z) &= \mu_Z\\ &= \mathrm{E}(\alpha_0 + \alpha_1 W)\\ &= \mathrm{E}(\alpha_0) + \mathrm{E}(\alpha_1 W)\\ &= \alpha_0 + \alpha_1 \mathrm{E}(W)\\ &= \alpha_0 + \alpha_1 \mu_W\end{aligned}$$

This can also be shown by using the fact that $z_i = \alpha_0 + \alpha_1 w_i$

$$\begin{aligned}
\mathrm{E}(Z) &= \mu_Z \\
&= \sum_i f(x_i) z_i \\
&= \sum_i f(w_i)(\alpha_0 + \alpha_1 w_i) \\
&= \sum_i f(w_i)\alpha_0 + \sum_i f(w_i)\alpha_1 w_i \\
&= \alpha_0 \sum_i f(w_i) + \alpha_1 \sum_i f(w_i) w_i \\
&= \alpha_0 + \alpha_1 \,\mathrm{E}(W) \\
&= \alpha_0 + \alpha_1 \mu_W
\end{aligned}$$

(b) Variance of Z.

$$\begin{aligned}
\mathrm{Var}(Z) &= \sigma_Z \\
&= \mathrm{Var}(\alpha_0 + \alpha_1 W) \\
&= \mathrm{Var}(\alpha_0) + \mathrm{Var}(\alpha_1 W) \\
&= 0 + \alpha_1^2 \,\mathrm{Var}(W) \\
&= \alpha_1^2 \sigma_W^2
\end{aligned}$$

Since $z_i = \alpha_0 + \alpha_1 w_i$, another way of arriving to the former result is shown here:

$$\begin{aligned}
\mathrm{Var}(Z) &= \sigma_Z \\
&= \sum_i f(x_i)(z_i - \mu_Z)^2 \\
&= \sum_i f(w_i)\left[(\alpha_0 + \alpha_1 w_i) - (\alpha_0 + \alpha_1 \mu_W)\right]^2 \\
&= \sum_i f(w_i)(\alpha_1 w_i - \alpha_1 \mu_W)^2 \\
&= \sum_i f(w_i)\alpha_1^2 (w_i - \mu_W)^2 \\
&= \alpha_1^2 \sum_i f(w_i)(w_i - \mu_W)^2 \\
&= \alpha_1^2 \sigma_W^2
\end{aligned}$$