Slides III - Complete Markets

Julio Garín
University of Georgia

Macroeconomic Theory II (Ph.D.)

Spring 2017

Outline

Uncertainty and Expected Utility Theory

- We usually don't have all information available about the future.
- We typically use "Expected Utility" Theory.
  - Von Neumann-Morgenstern preferences.
- Let's focus first in a case with two "states of the world".
- It's all about trade-offs.
  - Intertemporal elasticity of substitution (IES).
  - Coefficient of relative risk aversion (RRA).

Arrow-Debreu Model of State-Contingent Securities

- The baseline model of consumption under uncertainty.
  - What will it teach us?
- Agents are expected-utility maximizers.
- We will start with a simple two-period model.
  - Why?
  - Endowment economy.
    - Income at $t+1$, y_{t+1}, is uncertain.
- State-contingent (Arrow-Debreu securities) are available.
  - Pay one unit of output if state s occurs, otherwise pay zero.

Two-Period Problem With Only Two States

- Definitions:
  - Number of state-s securities that the agent holds.
  - Price paid for those securities.
  - Probability that state s occurs.
- There are "complete markets":
  - There are A-D securities for every state of the world.
    - This doesn't imply perfect consumption smoothing.
- We will also allow a non-contingent bond, b_{t+1}.

- We have to solve:

$$\max_{\substack{c_t,\, c_{t+1}(1),\, c_{t+1}(2), \\ B_{t+1}(1),\, B_{t+1}(2),\, b_{t+1}}} \quad u(c_t) + \beta\left[\pi_1 u(c_{t+1}(1)) + \pi_2 u(c_{t+1}(2))\right]$$

subject to:

$$c_t = y_t - b_{t+1} - P_t(1)B_{t+1}(1) - P_t(2)B_{t+1}(2)$$
$$c_{t+1}(1) = y_{t+1}(1) + B_{t+1}(1) + (1+r)b_{t+1}$$
$$c_{t+1}(2) = y_{t+1}(2) + B_{t+1}(2) + (1+r)b_{t+1}$$

Analyzing The First-Order Conditions

$$\frac{1}{1+r} = P_t(1) + P_t(2) \tag{1}$$

- What if this condition is not satisfied?
  - This would represent an *arbitrage* opportunity.
    - Market must drive prices such that the condition holds.
  - Non-contingent bonds are *redundant*.
    - They can be replicated by holding A-D securities.

$$u'(c_t) = (1+r)\beta\,\mathbb{E}_t\,u'(c_{t+1}) \tag{2}$$

- Euler equation under uncertainty.

- We can also show:

$$\frac{u'\left[c_{t+1}(1)\right]}{u'\left[c_{t+1}(2)\right]} = \frac{P_t(1)}{P_t(2)}\frac{\pi_2}{\pi_1} \tag{3}$$

  - If prices are *actuarially fair*, two marginal utilities are equal.
    - Consumption *would* be perfectly smoothed across states.

Economy With Heterogeneous Agents

- Two agents: Igor, i, and James, j.
- Receive different income streams.
  - *Idiosyncratic* risk.
- For some state s:

$$\frac{\beta \pi_s u' \left[c^k_{t+i}(s) \right]}{\lambda^k_t} = P_t(s) \quad \text{for } k = i, j \tag{4}$$

- Since they face the same price...

Economy With Heterogeneous Agents

$$\frac{u'\left[c^{j}_{t+i}(s)\right]}{u'\left[c^{i}_{t+i}(s)\right]} = \frac{\lambda^{j}_{t}}{\lambda^{i}_{t}} \tag{5}$$

- Lambdas are not state-dependent.
  - MUs are perfectly correlated across states.
  - State-contingent bonds allows for smoothing.
    - Insurance against *idiosyncratic* risk.
  - MUs can only change due to *aggregate* risk.
- How this related to the *representative agent* assumption?
  - If markets are complete, idiosyncratic risk will be insured.
    - Only aggregate risk remains.
    - Individuals will move up and down together.

Equilibrium Price Determination

- Assume that there are many agents, but they are all *identical*.
- Then prices must be such that in *equilibrium* agents consume their endowment.
  - In that case:
$$\frac{u'\left[y_{t+1}(1)\right]}{u'\left[y_{t+1}(2)\right]} = \frac{P_t(1)}{P_t(2)}\frac{\pi_2}{\pi_1}$$
  - If $y_{t+1}(1) \neq y_{t+1}(2)$ and utility isn't linear, prices are *not* actuarially fair.
- State-contingent bonds that deliver output when output is scarce are more expensive (relative to actuarially fair prices).
  - Idea at the foundation of all modern asset-pricing theory.

Full-Blown Infinite Horizon: Preliminaries

- *History* $s^t = [s_0, s_1, \ldots, s_t]$ keeps track of the sequence of states that have been realized.
- Households have von Neumann-Morganstern preferences.
- There are two alternative trading arrangements:
  1. Arrow-Debreu structure.
  2. Sequential trading (Radner).
     - Under some conditions, both are equivalent.
- Consumption allocation at time t will depend only on aggregate endowment.

Time-0 Trading

Time-0 Trading

- Agents buy and sell, at the beginning of time, a full set of state-contingent securities for each possible history and for each point in time.
- Probabilities:
  - $\pi_t(s^t)$ denotes unconditional probability of history s^t being realized.
    - Note that π_t varies through time.
  - $\pi_t(s^t|s^\tau)$ for $t > \tau$, denotes conditional probability.
- I households named $i = 1, \ldots, I$.
  - Each receives its own endowment, $y_t^i(s^t)$.
  - $q_t^0(s^t)$ is the time-0 price of a security that pays one unit of the endowment good at time t if history s^t is realized.

Time-0 Trading: Household Problem

- In equilibrium, prices are such that markets clear.
- The problem for household i is then

$$\max_{\{\{c_t^i(s^t)\}_{s^t}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \sum_{s^t} \beta^t u[c_t^i(s^t)] \pi_t(s^t)$$

subject to:

$$\sum_{t=0}^{\infty} \sum_{s^t} q_t^0(s^t) c_t^i(s^t) \leq \sum_{t=0}^{\infty} \sum_{s^t} q_t^0(s^t) y_t^i(s^t)$$

- Note that:
  - *Single* lifetime budget constraint.
  - We don't keep track of how many state-contingent claims the household purchases.
  - Consumption and income don't need to be equal to each other in every state.

Equilibrium

Definition

A price system is a sequence of functions $\left\{q_t^0(s^t)\right\}_{t=0}^{\infty}$*. An allocation is a list of sequences of functions* $c^i = \left\{c_t^i(s^t)\right\}_{t=0}^{\infty}$*, one for each* i*.*

Definition (Competitive Equilibrium)

A competitive equilibrium is a feasible allocation and price system such that, given the price system, the allocation solves each household's problem.

F.O.C.

- Let μ_i be the Lagrange multiplier on household's i's lifetime budget constraint.

$$\beta^t u' \left[c_t^i(s^t)\right] \pi_t(s^t) = \mu_i q_t^0(s^t)$$

  - We can show that, consistent with what we derived before, different household's MU are perfectly correlated.
- After inverting the MU and plugging into the aggregate feasibility constraint,

$$\sum_i u'^{-1} \left\{ u' \left[c_t^j(s^t)\right] \frac{\mu_i}{\mu_j} \right\} = \sum_i y_t^i(s^t)$$

  - $c_t^j(s^t)$ only depends on the current aggregate endowment.
    - All idiosyncratic risk is traded away.

Example 1: Risk Sharing

- Assume $u(c) = \frac{c^{1-\gamma}-1}{1-\gamma}$.
- Consumption of household i will be given by

$$c_t^i(s^t) = c_t^j(s^t)\left(\frac{\mu_i}{\mu_j}\right)^{-\frac{1}{\gamma}}$$

- And using the feasibility constraint we can derive

$$c_t^j(s^t) = \chi_j^{-1}\sum_i y_t^i(s^t)$$

where $\chi_j = \mu_j^{1/\gamma}\sum_i \mu_i^{-1/\gamma}$ gives the fraction of aggregate endowment that agent j consumes in each period.

Example 2: No Aggregate Uncertainty

- Two households.
  - Receive $y_t^1(s^t) = s_t$ and $y_t^2(s^t) = 1 - s_t$, where s_t is a r.v. taking on values in $[0, 1]$
- Since aggregate endowment is constant $c_t^i(s^t) = \bar{c}^i$.
- From the F.O.C. and the lifetime budget constraint we can show that:

$$\bar{c}^i = (1 - \beta) \sum_{t=0}^{\infty} \sum_{s^t} \beta^t \pi_t(s^t) y_t^i(s^t)$$

  - Is this familiar?
    - What if $\beta^{-1} = 1 + r$?
  - Individual consumptions will depend on the distribution of s_t.
  - Note that $\bar{c}^1 + \bar{c}^2 = 1$.

Asset Pricing

Asset Pricing

- Rearranging the F.O.C.

$$q_t^0(s^t) = \beta^t \frac{u'\left[c_t^i(s^t)\right]\pi_t(s^t)}{\mu_i}$$

  - Does it make sense?
- As before, negative relationship between price and equilibrium consumption.
  - People are willing to pay more for deliveries when aggregate endowment is scarce.
    - Expression in terms of household i is without loss of generality.

Applications

- We can use these prices to price all sort of *redundant* assets.
  - Just apply the price to the dividend/payment that the asset pays.
- Let $\{d_t(s^t)\}_{t=0}^{\infty}$ be the stream of claims on time t, history s^t consumption.
  - No arbitrage implies:

$$p_0^0(s_0) = \sum_{t=0}^{\infty} \sum_{s^t} q_t^0(s^t) d_t(s^t)$$

- Examples:
  1. Riskless consol.
  2. Riskless strips.

Tail Assets

- After stripping the first $\tau - 1$ periods, what is the time 0 value of the remaining dividend stream?

$$p_\tau^0(s^\tau) = \sum_{t \geq \tau} \sum_{s^t | s^\tau} q_t^0(s^t) d_t(s^t)$$

  - We can convert this to time τ, history s^τ units:

$$p_\tau^\tau(s^\tau) \equiv \frac{p_\tau^0(s^\tau)}{q_\tau^0(s^\tau)} = \sum_{t \geq \tau} \sum_{s^t | s^\tau} \frac{q_t^0(s^t)}{q_\tau^0(s^\tau)} d_t(s^t)$$

  - Defining $q_t^\tau \equiv q_t(s^t) / q_\tau^0(s^\tau)$ we obtain

$$p_\tau^\tau(s^\tau) = \sum_{t \geq \tau} \sum_{s^t | s^\tau} q_t^\tau(s^t) d_t(s^t)$$

    - Note what $q_t^\tau(s^t)$ really is.

One Period Returns and Stochastic Discount Factor

- What is the time τ price at history s^τ of a claim to a random payoff, $\omega(s_{\tau+1})$?

$$p_\tau^\tau(s^\tau) = \sum_{s_{\tau+1}} q_{\tau+1}^\tau(s^{\tau+1})\omega(s_{\tau+1})$$

- Using our definition of $q_t^\tau(s^t)$ we have that for any individual i

$$\begin{aligned} 1 &= \mathbb{E}_\tau \left[\beta \frac{u'(c_{\tau+1})}{u'(c_\tau)} R_{\tau+1}\right] \\ &= \mathbb{E}_\tau \left[m_{\tau+1} R_{\tau+1}\right] \end{aligned}$$

where $R_{\tau+1}$ is the one-period gross return on the asset and $m_{\tau+1}$ is a *stochastic discount factor*.

Sequential Trading

Sequential Trading

- Arrow securities.
- Tildes denote equilibrium variables from this environment.
- New definitions:
  - Claims to time-t endowment households bring to t.
  - Price of an Arrow security.
- The budget constraint is given by:

$$\tilde{c}_t^i(s^t) + \sum_{s_{t+1}} \tilde{a}_{t+1}^i(s^{t+1})\tilde{Q}_t^i(s_{t+1}|s^t) \leq \tilde{y}_t^i(s^t) + \tilde{a}_t^i(s^t)$$

Sequential Trading: Borrowing Limit

- Before there was a 'payment clearance system'.
- We still need to make sure that everyone lives within their means.
  - We impose the no-Ponzi scheme:

$$-\tilde{a}_{t+1}^{i}(s^{t+1}) \leq \sum_{\tau=t+1}^{\infty} \sum_{s^{\tau}|s^{t}} q_{\tau}^{t+1}(s^{\tau}) y_{\tau}^{i}(s^{\tau}) \equiv A_{t}^{i}(s^{t})$$

    - Note that prices are not in terms of $\tilde{Q}$s.
  - This is a "natural borrowing limit".

Sequential Trading: Household Problem

- The problem for household i is then

$$\max_{c_t^i(s^t), \{\tilde{a}_{t+1}^i(s_{t+1|s^t})\}} \sum_{t=0}^{\infty} \sum_{s^t} \beta^t u[c_t^i(s^t)] \pi_t(s^t)$$

subject to:

$$\tilde{c}_t^i(s^t) + \sum_{s_{t+1}} \tilde{a}_{t+1}^i(s^{t+1}) \tilde{Q}_t^i(s_{t+1}|s^t) \leq \tilde{y}_t^i(s^t) + \tilde{a}_t^i(s^t) \quad (6)$$

$$-\tilde{a}_{t+1}^i(s^{t+1}) \leq A_t^i(s^t) \quad (7)$$

  - Let $\eta_t^i(s^t)$ and $\nu_t^i(s^t)$ be the multipliers on (6) and (7).

F.O.C.

- Combing the F.O.C.s and using the Inada condition on utility, we obtain

$$\tilde{Q}_t^i(s_{t+1}|s^t) = \beta \frac{u'\left[\tilde{c}_{t+1}^i(s^{t+1})\right]}{u'\left[\tilde{c}_t^i(s^t)\right]} \frac{\pi_{t+1}(s^{t+1})}{\pi_t(s^t)}$$

  - From time-0 endowment:

$$q_t^0(s^t) = \beta \frac{u'\left[c_t^i(s^t)\right] \pi(s^t)}{\mu_i}$$

    and

$$q_t^{t+1}(s^t) \equiv \frac{q_{t+1}^0(s^{t+1})}{q_t^0(s^t)}$$

- What if $\tilde{Q}_t^i(s_{t+1}|s^t) = q_t^{t+1}(s^{t+1})$?

Equivalence

- Define the household i's current and future net claims under time-0 equilibrium:

$$\Upsilon_t^i(s^t) = \sum_{\tau=t}^{\infty} \sum_{s^\tau | s^t} q_\tau^t(s^\tau) \left[c_\tau^i(s^\tau) - y_\tau^i(s^\tau) \right]$$

  - We can show that if $\forall\ i, a_0^i(s_0) = 0$, then the lifetime budget constraints are the same.
  - Also, $\tilde{a}_t^i(s^t) = \Upsilon_t^i(s^t)$.
    - Therefore, equilibrium allocation are the same.
  - Welfare theorems apply, so Pareto optimum is achieved.
    - Intuition?

Markov and Recursivity

- So far, probabilities could depend on entire past history.
  - Pricing kernels and wealth distributions both depend on history s^t.
    - Both are time-varying functions.
- Now let's assume that states follow a Markov process.
  - If the same is true for endowment, we will have $y_t^i(s^t) = y_t^i(s_t)$.
    - The problem takes a recursive structure.
- We will have a new equilibrium concept.

History Independence and Equilibrium Outcomes

- We have

$$\sum_i u'^{-1}\left\{u'\left[c_t^j(s^t)\right]\frac{\mu_i}{\mu_j}\right\} = \sum_i y_t^i(s^t)$$

  - Given the Markov property, we can express $c_t^i(s^t) = \bar{c}^i(s_t)$
    - $\bar{c}()$ is a time-invariant mapping that depends only on current state.
  - Therefore, we can rewrite the Euler:

$$\tilde{Q}_t^i(s_{t+1}|s^t) = \beta\frac{u'\left[\bar{c}^i(s_{t+1})\right]}{u'\left[\bar{c}^i(s_t)\right]}\pi(s_{t+1}|s_t)$$

- This *history independence* extends to the household's wealth level

$$\mathrm{Y}_t^i(s^t) = \bar{\mathrm{Y}}_t^i(s_t)$$

  - Recall what Y^i was.
  - Why doesn't it depend on past endowment realizations?
    - Household already insured himself against previous endowment realizations.