Slides II - Dynamic Programming

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Review of Two-Period Intertemporal Optimization

Model Overview

- Agents:
  - *Forward looking*.
  - Rational and their behavior is optimizing.
- Preferences:
  - Lifetime utility:

$$\mathcal{U} = u(c) + \beta u(c')$$

  - Two periods is unrealistic but useful.
  - Slope of the indifference curve.
  - If $u_{cc} < 0$:
    - Marginal utility is decreasing in consumption.
    - Quasi-concave utility implies convex preferences.
    - Agent would like to smooth consumption.
- 4 Technology Scenarios:
  1. Endowment economy, no assets.
  2. Endowment economy, riskless, interest-bearing asset.
  3. "Robinson Crusoe" economy.
  4. Productive capital and riskless asset.

Model I: Endowment Economy, no Assets, and no Storage

- Assume no assets and no storage.
  - What is consumption equal to?
- Technology.
  - Production.
    - For now, we will look at an endowment economy.
    - Stream of income given by y and y'.
  - How can output be transferred between periods?
    - We start with a one-period riskless bond.

Model II: Endowment Economy with Riskless Asset

- Now households have access to a riskless asset, a, that pays r.
  - Amount borrowed limited by $\underline{a}$.
- Household's problem given by:

$$\max_{\{c,c',a\}} \mathcal{U} = u(c) + \beta u(c')$$

subject to:

$$a = y - c \tag{1}$$
$$c' = (1+r)a + y' \tag{2}$$
$$a \geq \underline{a} \tag{3}$$

- We can solve using Lagrangian methods.
  - Importance of the Euler equation.

Model III: A Robinson Crusoe Economy

- Now, instead of a riskless asset, households can move resources forward in time by buying productive capital.
  - Increasing and concave production function: $y = f(k)$.
- Household's problem becomes:

$$\max_{\{c,c',k'\}} \mathcal{U} = u(c) + \beta u(c')$$

subject to:

$$c + k' = f(k) + (1-\delta)k \tag{4}$$

$$c' = f(k') + (1-\delta)k' \tag{5}$$

$$k \text{ is given} \tag{6}$$

- Again, we can use Lagrangian methods and obtain the Euler.
  - Now the rate of return is $f_{k'} + 1 - \delta$.

Model IV: Productive Capital and Riskless Asset

- The agent faces an asset allocation problem.
- The problem is given by:

$$\max_{\{c,c',a',k'\}} \mathcal{U} = u(c) + \beta u(c')$$

subject to:

$$c + k' + a' = f(k) + (1-\delta)k \quad (7)$$

$$c' = f(k') + (1-\delta)k' + (1+r)a' \quad (8)$$

$$k \text{ is given} \quad (9)$$

- Using Lagrangian methods, in addition to the standard Euler we obtain.
  - $f_{k'} + 1 - \delta = 1 + r$.

Dynamic Programming

Why “Dynamic Programming”?

[...]
What title, what name, could I choose? In the first place I was interested in planning, in decision making, in thinking. But planning, is not a good word for various reasons. I decided therefore to use the word ‘programming’. I wanted to get across the idea that this was dynamic, this was multistage, this was time-varying–I thought, let’s kill two birds with one stone. Let’s take a word that has an absolutely precise meaning, namely ‘dynamic’, in the classical physical sense. It also has a very interesting property as an adjective, and that is it’s impossible to use the word ‘dynamic’ in a pejorative sense. Try thinking of some combination that will possibly give it a pejorative meaning. It’s impossible. Thus, I thought “dynamic programming” was a good name. It was something not even a Congressman could object to. So I used it as an umbrella for my activities.
– *Richard Bellman*

Basics

- Ultimately we would like to think about multi-period models... or even infinite horizon models.
  - Why?
- Suppose we have the following problem for a household:

$$\max_{\{c_t, b_t\}_{t=0}^{\infty}} \mathcal{U}_0 = \sum_{t=0}^{\infty} \beta^t u(c_t)$$

subject to:

$$c_t + \frac{b_{t+1}}{1+r} = b_t + y_t \quad \text{for } t = 0, 1, 2, \ldots$$
$$b_0 \geq 0 \text{ given}$$

- Now bonds are *discounted* bonds.

Why Do We Need Another Tool?

- For a rational, forward-looking household, consumption/saving decisions are intertemporally related.
- We could use Lagrangian methods to solve this intertemporal optimization problem.
  - System of many non-linear equations in many unknowns.
    - Hard, but feasible.
- Dynamic programming helps us to transform this problem into something more tractable, especially for stochastic models.
- We will start with a three-period problem.
  - Think about problems *recursively*.
  - Introduce the lingo.

Three-Period Consumption/Saving Problem

- The problem is:

$$\max_{\{c_{t+j}, b_{t+j}+1\}_{j=0}^{3}} \mathcal{U}_0 = u(c_t) + \beta u(c_{t+1}) + \beta^2 u(c_{t+2})$$

subject to:

$$c_t + \frac{b_{t+1}}{1+r} = b_t + y_t$$

$$c_{t+1} + \frac{b_{t+2}}{1+r} = b_{t+1} + y_{t+1}$$

$$c_{t+2} + \frac{b_{t+3}}{1+r} = b_{t+2} + y_{t+2}$$

$$b_t \geq 0 \text{ given}$$

- Let's assume for now that $y_{t+j} = 0$ for $j = 0, 1, 2$.

Solving the Three-Period Problem: $t + 2$

- We solve the problem *backwards*.
  - Today's decisions are not independent of future outcomes.
  - The *Principle of Optimality*.
- At period $t + 2$ the problem is:

$$\max_{\{c_{t+2}, b_{t+3}\}} \mathcal{U}_0 = u(c_{t+2})$$

subject to:

$$c_{t+2} + \frac{b_{t+3}}{1+r} = b_{t+2}$$

- Since $u' > 0 \Rightarrow b^*_{t+3} = 0$ and

$$c^*_{t+2} \equiv f_{t+2}(b_{t+2}) = b_{t+2}$$

- Given the *state variable*, the *policy function*, f, tells the optimal value for the *choice variable*.

Solving the Three-Period Problem: Value Function

- We can define $V_{t+2}(b_{t+2}) \equiv \mathcal{U}^*_{t+2}$.
- The *value function*, $V_{t+2}(b_{t+2})$, is defined by the following *functional equation*:

$$V_{t+2}(b_{t+2}) = \max_{\{c_{t+2}, b_{t+3}\}} u(c_{t+2})$$

subject to:

$$c_{t+2} + \frac{b_{t+3}}{1+r} = b_{t+2}$$

- The *transition equation* maps, for a given value of the choice variable, the state variable in one period into the next period's state variable.
- Note what $V'(b_{t+2})$ is equal to.
    - We will come back to this.

Solving the Three-Period Problem: $t+1$

- Let's keep working backwards:

$$\max_{\{c_{t+1}, b_{t+2}\}} \mathcal{U}_{t+1} = u(c_{t+1}) + \beta V_{t+2}(b_{t+2})$$

subject to:

$$c_{t+1} + \frac{b_{t+2}}{1+r} = b_{t+1}$$

- The *continuation value* is $V_{t+2}(b_{t+2})$ and is taken as given.
  - Decisions today affect future via changes in state variables, not via changes in future policy functions.
    - This *time consistency* of decision rules is known as the Principle of Optimality.
- We can transform this problem into an unconstrained one.
  - What is the *control* variable? What is the state variable?

- The F.O.C. for the unconstrained problem is:

$$u'\left(b_{t+1} - \frac{b_{t+2}}{1+r}\right) = \beta(1+r)V'_{t+2}(b_{t+2})$$

  - Intuition?

- Given b_{t+1} we have

$$b^*_{t+2} = g_{t+1}(b_{t+1})$$

  and from there we can solve for the optimal consumption from the budget constraint:

$$c^*_{t+1} = f_{t+1}(b_{t+1})$$

- Note that these policy functions are *time-dependent*.
- Write down the value function representation.

Solving the Three-Period Problem: t

$$\max_{c_t, b_{t+1}} \mathcal{U}_0 = u(c_t) + \beta V_{t+1}(b_{t+1})$$

subject to:

$$c_t + \frac{b_{t+1}}{1+r} = b_t$$

- The F.O.C. looks familiar.
  - From there we obtain b^*_{t+1} and c^*_t.
- While we solve backwards in time, we now can use the optimal policy functions to move forward, starting from an initial b_t.

Method Comparison: Lagrange vs. Dynamic Programming

We will compare the methods with a very simple utility model.

Simple Model Method Comparison: Lagrange Method

Let's solve:

$$\max_{\{c_0, c_1, c_2, b_1, b_2\}} \mathcal{U}_0 = \ln(c_0) + \beta \ln(c_1) + \beta^2 \ln(c_2)$$

subject to:

$$c_0 + \frac{b_1}{1+r} = b_0$$

$$c_1 + \frac{b_2}{1+r} = b_1$$

$$c_2 + \frac{b_3}{1+r} = b_2$$

$$b_0 \geq 0 \text{ given}$$

- Note that the solution is in recursive form.

Simple Model Method Comparison: Dynamic Programming

- As before, we use backward induction.
  - We know what will happen in period 2.
- We start with

$$\max_{\{c_1, b_2\}} \ln(c_1) + \beta V_2(b_2)$$

subject to:

$$c_1 + \frac{b_2}{1+r} = b_1$$

- From there, let's go to period 1 and 0.
- Clearly, $\tilde{f}_i = f_i$ and $\tilde{g}_i = g_i$ for $i = 0, 1$.
  - Consistent with Principle of Optimality.

From Period t to $t+T$

If $T \to \infty$, the problem

$$V_t(b_t) = \sum_{i=0}^{T} \beta^i u(c^*_{t+i})$$
$$= \max_{c_t, b_{t+1}} u(c_t) + \beta V_{t+1}(b_{t+1})$$

subject to:

$$c_t + \frac{b_{t+1}}{1+r} = b_t$$

can be expressed as

$$V(b_t) = \max_{c_t, b_{t+1}} u(c_t) + \beta V(b_{t+1})$$

subject to:

$$c_t + \frac{b_{t+1}}{1+r} = b_t$$

- Value functions and policy functions will be *time-autonomous*.

Transversality Condition

- In the finite horizon we implicity ruled out dying with debt.
  - Now we have a similar condition: *transversality condition*.
- Let's put the income process back into the problem.
- After some work, we find that the condition is given by

$$\lim_{n\to\infty}\left(\frac{1}{1+r}\right)^n b_{t+n} = 0$$

  - A relatively weak condition.
- If we let $n \to \infty$ and we impose the transversality condition we have

$$\sum_{i=0}^{\infty}\left(\frac{1}{1+r}\right)^i c_{t+i} = b_t + \sum_{i=0}^{\infty}\left(\frac{1}{1+r}\right)^i y_{t+i}$$

A General Version

- We have the problem

$$\max_{\{u_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t r(x_t, u_t)$$

subject to:

$$x_{t+1} = g(x_t, u_t)$$
$$x_0 \text{ given}$$

  - $r(x_t, u_t)$ is a concave *return* or *payoff* function.
  - What are the state and the control variables?
  - Condition on transition equation:
    - It must represent a convex and compact set.
- Mapping a known model into this problem.

- If we didn't know dynamic programming we would have to find the *optimal infinite sequence* of choice variables.
  - What about a stochastic setting?
  - If we could solve for that sequence, then we could obtain $V(x_0)$.
- Dynamic programming offers an alternative:
  - We seek the optimal *time-invariant* policy function.
    - With the transition equation, this gives us the sequence $\{u_t^*\}_{t=0}^{\infty}$.

- If we knew $V(x)$, we could find $h(x)$ by solving:

$$\max_{u,x'} r(x,u) + \beta V(x')$$

subject to:

$$x' = g(x,u)$$

  - For each x, compute $h(x)$.
  - No time subscripts.
- If we knew $h(x)$ we could solve the following for $V(x)$:

$$V(x) = r(x,h(x)) + \beta V(g(x,h(x)))$$

  - For each x, compute $V(x)$.
- However...
  - We don't know $h(x)$ nor $V(x)$!
    - Have to solve for them simultaneously.

- We will solve for $h(x)$ and $V(x)$ *simultaneously*, by solving the Bellman equation:

$$V(x) = \max_{u} r(x, u) + \beta V(g(x, u))$$

  - How do we know that the recursive solution to this problem is the same as the solution to the sequential problem?
  - Under appropriate assumption the operator in the right side is a *contraction mapping*.
    - The V that solves the Bellman equation is a fixed point of the operator.
- When is an operator a contraction mapping?
  - Blackwell's sufficient conditions.
  - Why do we care?
    - When operators are contractions, repeated application of the operator will converge to the fixed point.

Properties of the Bellman Equation

- Under 'standard' assumptions:
  1. There is a unique solution, V, to the Bellman equation and it is strictly concave.
  2. As $j \to \infty$, V is approached in the limit by iterating on

$$V_{j+1}(x) = \max_u r(x,u) + \beta V_j(g(x,u))$$

  3. There is a unique and time-invariant policy function.
  4. Benveniste-Scheinkman envelope condition.
     - $V(x)$ is differentiable.

Stochastic Dynamic Programming

- Not too much changes.
  - Add expectation operator.
  - Might add another state variable.
    - The stochastic element may become a state variable.
- Example: stochastic growth model with persistent productivity shocks:

$$V(k, z) = \max_{\{c, k'\}} u(c) + \beta \, \mathbb{E}_{z'|z} \, V(k', z')$$

subject to:

$$k' + c = zf(k)$$

  - z would be a state variable even if it is i.i.d.
  - What if we had a growth model with stochastic and i.i.d. depreciation?

Solution Techniques: Value Function Iteration

- Inspired on 2.
  1. Make an initial guess about V, call it V_i.
  2. Use this guess on the r.h.s. of the Bellman equation.
  3. Maximizing on the r.h.s. yields new V, call it V_{i+1}
  4. Use this new V_{i+1} as the V_i on the r.h.s. in the next iteration.
  5. Iterate until V_i and the resulting V_{i+1} are nearly equal.
- This rarely works in closed form.
- When solving numerically, the key is how to represent functions like V.

Solution Techniques: Guess and Verify

- Guess functional form and verify that it solves the Bellman equation.
- Only works with problems that have a closed-form solution.

Solution Techniques: Policy Function Iteration

- Also known as "Howard's Improvement Algorithm".
  - 'Opposite' to VFI.

1. Start with a guess for the policy function, $h(x)$.
2. Calculate the value of following $h(x)$ forever.
3. Calculate new policy function by maximizing r.h.s. of the Bellman with the V from the previous step.
4. Iterate until convergence.

- There are other techniques as well.