Slides I - Math Preliminaries

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Outline

1. Difference equations.
    1.1 First-order deterministic linear difference equations.
    1.2 System of difference equations.
    1.3 Second-order difference equations.
    1.4 First-order stochastic linear difference equations.
2. Markov processes.
    2.1 Unconditional Probability Distributions.
    2.2 Ergodic distributions.

First-Order Deterministic Linear Difference Equations

- We'll work with the following difference equation:

$$x_t = ax_{t-1} + y_t.$$

  - For now, y_t will be constant.
  - The steady state.
- Importance of the slope.
  - If $0 < a < 1$, x_t approaches its steady state in a *monotonic* fashion.
  - If $-1 < a < 0$, x_t approaches its steady state in a *oscillatory* fashion.
  - If $|a| > 1$, x_t diverges.
  - If $|a| = 1$, x_t either oscillates endlessly, or grows linearly.
- Example: unemployment rate.

Solution

- What does it mean to have a 'solution'?
- Iterative method.
  - Backward solution.
  - Forward solution.

$$u_t = bu_{t+1} + v_t.$$

    - Relationship between $|a|$ and $|b|$.
    - Example: Price dividend-paying stock

$$p_t = \frac{1}{1+r}(p_{t+1} + d_{t+1})$$

    - Note that if $d_t = \bar{d}\ \forall t$, then $p_t = \bar{d}/r$..

System of Difference Equations

- Consider the variables p_t and y_t, whose time paths depend on each other:

$$p_t = ap_{t-1} + by_{t-1} + m$$
$$y_t = cp_{t-1} + dy_{t-1} + g.$$

- We can use matrix notation and express it as

$$n_t = An_{t-1} + x.$$

Solution

- What if the off-diagonal elements are zero?
  - It turns out that we can transform the system and do that.

$$H^{-1}n_t = \Lambda H^{-1}n_{t-1} + H^{-1}x$$

or

$$u_t = \Lambda u_{t-1} + s.$$

where $\Lambda = H^{-1}AH$.

- If A is a nonsingular matrix s.t. $H^{-1}AH = \Lambda$, then Λ is a matrix with the two eigenvalues of A on the diagonal and H is a matrix with the two eigenvectors as columns.
  - We know how to solve that.
  - From u_t to n_t.
- The stability will depend on the properties of the eigenvalues of the matrix A.

Second-Order Difference Equations

- Suppose that now the equation to solve is:

$$x_t = ax_{t-1} + bx_{t-2} + m$$

- We can define the variable $y_t = x_{t-1}$ and re-write as a system of two FODE:

$$\begin{bmatrix} x_t \\ y_t \end{bmatrix} = \begin{bmatrix} a & b \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_{t-1} \\ y_{t-1} \end{bmatrix} + \begin{bmatrix} m \\ 0 \end{bmatrix}.$$

- How do we solve this?
- Just as before we needed two starting values for p_0 and y_0, here we will need x_1 and x_0.

Time Series and Stochastic Processes

- We will frequently want to be able to deal with stochastic processes.
  - What is a stochastic process?
  - Examples?
- We have essentially two ways of representing them:
  1. As FOSLDE.
  2. As Markov chains.

First-Order Stochastic Linear Difference Equations

- We have basically already seen these:

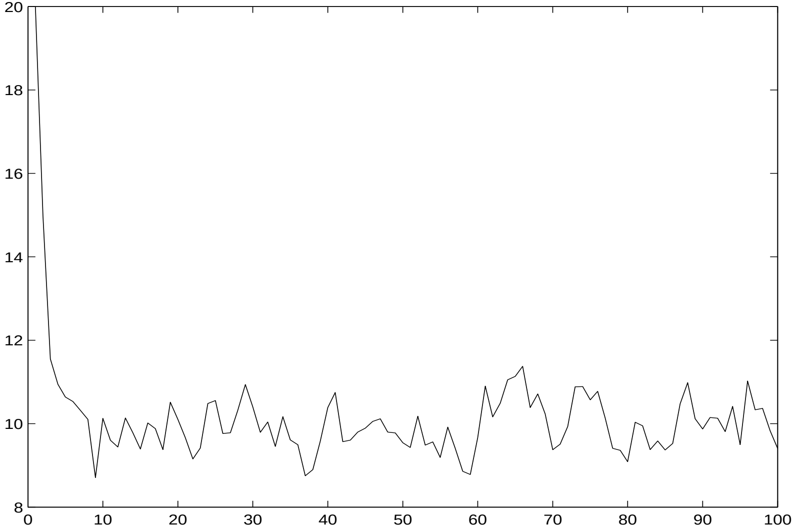
$$x_t = ax_{t-1} + y_t.$$

- Now "shocks" from the (i.i.d.) r.v. y_t will prevent x_t from converging to a constant steady state.
- We will have another notion of stability:

A stochastic process $\{x_t\}$ is <u>covariance stationary</u> if

1. $\mathbb{E}[x_t] = \mu, \ \forall t$.
2. $\mathbb{E}[(x_t - \mu)^2] = \sigma, \ \forall t$.
3. $\mathbb{E}[(x_{t+j} - \mu)(x_t - \mu)] = \gamma_j, \ \forall t$ and j.

Stationarity



Markov Process

- We will consider discrete-state markov chains.
- The "Markov property".
  - Probability distribution over possible values in one period depends only on the previous period's value.
    - 'Memoryless process.'
- Why may a Markov process be useful?

Markov Chain

A time-invariant discrete-state Markov chain consists of:

- A vector $\bar{x}$ of the possible values that x_t can have.
- A transition (or stochastic) matrix $\mathbb{P}$.
  - $\sum_{j=1}^{n} \mathbb{P}_{i,j} = 1$.
- A vector π_0 giving the initial probability distribution.
  - $\sum_{i=1}^{n} \pi_{0,i} = 1$.

Unconditional and Conditional Probability Distributions

- Let π_t be the vector whose n entries denote the probability that x_t is in each of the n states at time t.
  - We have that $\pi_1' = \pi_0' \mathbb{P}$.
    - $\Rightarrow \pi_k' = \pi_0' \mathbb{P}^k$.
- Unconditional probability distributions evolve according to

$$\pi_{t+1}' = \pi_t' \mathbb{P} .$$

- More generally

$$Pr(x_{t+k} = \bar{x}_j | x_t = \bar{x}_i) = (\mathbb{P}^k)_{ij}.$$

- Example.

Stationary Distribution

- An unconditional probability distribution is called stationary or invariant if it satisfies $\pi_{t+1} = \pi_t$ so that

$$\pi' = \pi' \mathbb{P}.$$

  - Note that this means that π is an eigenvector associated with a unit eigenvalue of $\mathbb{P}$.
    - We are guaranteed that there will be at least one unit eigenvalue.
- We can find the stationary distribution:
  1. By finding the eigenvector associated with the unit eigenvalue of $\mathbb{P}$ and normalizing.
  2. Solving $(\mathbb{P}' - \mathbb{I})\pi = 0$ or $\pi'(\mathbb{I} - \mathbb{P}) = 0$.

Uniqueness

- Does a Markov chain always have the property that

$$\lim_{t \to \infty} \pi_t = \pi_\infty = \pi.$$

for an arbitrary π_0?

  - Nope. Example.

- If $\mathbb{P}$ is <u>regular</u> or $\mathbb{P}^k_{i,j} > 0 \; \forall \; (i, j)$ then $\mathbb{P}$ has a unique stationary distribution and the process is asymptotically stationary.

Conditional and Unconditional Expectations

- *Conditional* expectation of x_t?
- *Unconditional* expectation at $t + k$?
- Law of iterated expectations relates conditional to unconditional.