Macroeconomic Theory II
Julio Garín
Spring 2017
Problem Set 3

Due Date: Beginning of class on Tuesday, February 28.
Instructions: Use LaTeX.

1. **Deterministic Endowment Economy.** In this pure exchange economy there exists a single, non-storable consumption good, c_t, and there are two infinitely lived individuals indexed by i, whose lifetime discounted utility is given by,

$$U(c^i) = \sum_{t=0}^{\infty} \beta^t \ln(c_t^i)$$

where $\beta \in (0,1)$ is the discount factor. Endowment of agent i at time t is given by e_t^i. There is an Arrow-Debreu market structure. At period 0, before endowments are received and consumption is realized, the two agents meet and trade contracts for all future dates. Let p_t^0 denote the price at time 0 of one unit of consumption to be delivered in period t, in terms of the numeraire good. Assume the market is competitive. After time-0 trade has happened, markets close and in all future dates the only thing that happens is that agents meet and deliver of the consumption good as stipulated by the contract.

   (a) Define the competitive equilibrium.

   (b) Set up the agent's problem and obtain the first order conditions. Show that the ratio of marginal utilities will be constant over time.

   (c) Show that time t consumption for agent 2 only depends on aggregate endowment.

   From now on assume the endowment streams, $e^i = \{e_t^i\}_{t=0}^{t=\infty}$, are given by:

$$e_t^1 = \begin{cases} 2 & \text{if } t \text{ is even} \\ 0 & \text{if } t \text{ is odd} \end{cases}$$

   and

$$e_t^2 = \begin{cases} 0 & \text{if } t \text{ is even} \\ 2 & \text{if } t \text{ is odd.} \end{cases}$$

   (d) Solve for the equilibrium allocations.

   (e) Do agents experience fluctuations in their consumption? Which agent enjoys a higher level of consumption? Justify.

   (f) Solve for the equilibrium prices. Make the time-0 consumption the numeraire good. Are prices increasing or declining over time? Why is that?

   (g) Calculate their lifetime discounted utility and show that both agents benefit from trade.

2. **State Contingent Claims.** Consider the following household problem:

$$\max_{c_t, c_{t+1}(1), c_{t+1}(2), b_{t+1}, B_{t+1}(1), B_{t+1}(2)} V_t = u(c_t) + \beta \{\pi_1 u[c_{t+1}(1)] + \pi_2 u[c_{t+1}(2)]\}$$

subject to:

$$y_t = c_t + \frac{1}{1+r_t} b_{t+1} + p_1 B_{t+1}(1) + p_2 B_{t+1}(2) \quad (1)$$

$$c_{t+1}(1) = y_{t+1}(1) + b_{t+1} + B_{t+1}(1) \quad (2)$$

$$c_{t+1}(2) = y_{t+1}(2) + b_{t+1} + B_{t+1}(2) \quad (3)$$

Assume the functional form $u(c) = \ln c$.

(a) Solve for the optimal values $c_t^*, c_{t+1}^*(1)$, and $c_{t+1}^*(2)$.

(b) As we showed in class, the availability of the two contingent bonds makes the non-contingent bond, b_{t+1}, redundant. Impose $b_{t+1}^* = 0$ and solve for the optimal values $B_{t+1}^*(1)$ and $B_{t+1}^*(2)$. If $y_{t+1}(1) < y_{t+1}(2)$ and prices are exogenously assumed to be actuarially fair, then what do we know about the size of $B_{t+1}^*(1)$ relative to $B_{t+1}^*(2)$?

(c) Now solve for the equilibrium prices p_1, p_2, and r_t when all agents have the same preferences and endowment streams. If $y_{t+1}(1) < y_{t+1}(2)$, are contingent claims prices actuarially fair? Why or why not?

(d) Suppose now that there are two agents in the economy. Both have $y_t = 1$. In the second period, the incomes of the first agent in the two states are $y_{t+1}^1(1) = 1$ and $y_{t+1}^1(2) = 2$, while for agent two $y_{t+1}^2(1) = 2$ and $y_{t+1}^2(2) = 1$. The two states are equally likely. What are the equilibrium prices and allocations? Discuss intuitively the results, referring to the roles of idiosyncratic and aggregate risk.

3. **Equivalence Between an Arrow-Debreu-McKenzie and a Radner Economy.** The economy is populated by individuals indexed by $j = 1, \ldots, J$. At each date there are a finite number of states of the world, $\mathbf{S}$. $s_t \in \{1, \ldots, \mathbf{S}\}$ is the current state (an event). The sequence of interest rates is given by $\{R_t\}$ and individual's j effective wealth is given by

$$x_0^j = (1 + R_0) a_0^j + h_0^j$$

where $(1 + R_0) a_0^j$ denotes time-0 financial wealth and h_0^j represents the present value of labor income. The latter can be written explicitly as

$$h_0^j = \sum_t \sum_{s^t} \frac{q(s^t)}{q_0} w(s^t) \left[l^j(s^t) - T^j(s^t) \right]$$

where $q(s^t)/q_0$ is the relative price of period-t consumption in terms of period-0 goods, $w(s^t)$ is the wage at time t and state s^t, and $l^j(s^t)$ and $T^j(s^t)$ represent, respectively, labor and taxes for individual j at time t and state s^t. Discounted expected utility for individual j at period t and state s^t depends on consumption, $c^j(s^t)$, and leisure $z^j(s^t)$:

$$\sum_t \sum_{s^t} \beta^t \pi(s^t) U\left[c^j(s^t), z^j(s^t)\right].$$

For now, let's assume the utility is separable in consumption and leisure:

$$U\left[c^j(s^t), z^j(s^t)\right] = u\left(c^j(s^t)\right) + v\left(z^j(s^t)\right).$$

(a) Set up the problem assuming an Arrow-Debreu economy. Derive individual's j first order conditions with respect to consumption both at time 0 and time t (recall that at $t = 0$, s_0 is observed so $\pi(s_0) = 1$).

(b) Assume CRRA utility with $\gamma > 0$ representing the coefficient of relative risk aversion. Show that consumption at time 0 depends only on aggregates and that individual consumption is linear in effective wealth.

(c) Show that all agents have identical state-contingent consumption growth rates. How might this justify assuming a representative agent?

In the Radner economy, there is a sequence of spot markets where people can trade in a complete set of Arrow securities for the next period. A state $s_{t+1} = k$ Arrow security pays one unit of the consumption good next period if k occurs; it pays nothing if k does not occur. The price of a date t Arrow security that pays off in state s_{t+1} is $v(s_{t+1}, s^t)$. Let $B_j(s_{t+1}, s^t)$ be the number of such securities held at t by agent j. The agent faces the sequence of budget constraints

$$c^j(s^t) + \sum_{s_{t+1}} v(s_{t+1}, s^t) B^j(s_{t+1}, s^t) = x^j(s^t) + B^j(s^t).$$

(d) Let $Q(s^t)$ be the date-t state contingent Lagrange multiplier. Compare the first order condition with respect to consumption of both environments (Arrow-Debreu and Radner economy). Interpret the first order condition with respect to the state-contingent bond.

(e) Show that the sequential budget constraint can be expressed as the intertemporal budget constraint from the Arrow-Debreu world. (Hint: use the recursive nature of the sequential budget constraint.) What is the implication of this result for the equilibrium allocations under both trading arrangements?

4. **Term Structure of Interest Rates.** (LS 3rd Ed., Chapter 8, Problem 8.2.) Consider an economy with a single consumer. There is one good in the economy, which arrives in the form of an exogenous endowment obeying[1]

$$y_{t+1} = \lambda_{t+1} y_t,$$

where y_t is the endowment at time t and λ_{t+1} is governed by a two-state Markov chain with transition matrix

$$P = \begin{bmatrix} p_{11} & 1 - p_{11} \\ 1 - p_{22} & p_{22} \end{bmatrix}$$

and initial distribution $\pi_\lambda = \begin{bmatrix} \pi_0 & 1 - \pi_0 \end{bmatrix}$. The value of λ_t is given $\bar{\lambda}_1 = 0.98$ in state 1 and $\bar{\lambda}_2 = 1.03$ in state 2. Assume that the histories of y_s, λ_s up to t are observed at time t. The consumer has endowment process $\{y_t\}$ and has preferences over consumption streams that are ordered by

$$\mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t u(c_t)$$

where $\beta \in (0, 1)$ and $u(c) = \frac{c^{1-\gamma}}{1-\gamma}$, where $\gamma \geq 1$.

(a) Define a competitive equilibrium, being careful to name all of the objects of which it consists.

(b) Explain how to compute a competitive equilibrium.

For the remainder of this problem, suppose that $p_{11} = .8, p_{22} = .85, \pi_0 = .5, \beta = .96$, and $\gamma = 2$. Suppose that the economy begins with $\lambda_0 = .98$ and $y_0 = 1$.

(c) Compute the (unconditional) average growth rate of consumption, computed before having observed λ_0.

[1]Such a specification was made by Mehra and Prescott (1985).

(d) Compute the time-0 prices of three risk-free discount bonds, in particular, those promising to pay one unit of time-j consumption for $j = 0, 1, 2$, respectively.

(e) Compute the time-0 prices of three bonds, in particular, ones promising to pay one unit of time-j consumption contingent on $\lambda_j = \bar{\lambda}_1$ for $j = 0, 1, 2$, respectively.

(f) Compute the time-0 prices of three bonds, in particular, ones promising to pay one unit of time-j consumption contingent on $\lambda_j = \bar{\lambda}_2$ for $j = 0, 1, 2$, respectively.

(g) Compare the prices that you computed in Parts 4d, 4e, and 4f.