Macroeconomic Theory II
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Problem Set 2

Due Date: Beginning of class on Tuesday, February 14.
Instructions: Use LaTeX. Label all plots, series, and axes accordingly.

1. **Riskless bonds**. Suppose the household's problem is given by

$$\max_{c_t, b_{t+1}, c_{t+1}} \quad V_t = u\left(c_t\right) + \beta u\left(c_{t+1}\right)$$

subject to:

$$c_t = y_t - b_{t+1}$$
$$c_{t+1} = y_{t+1} + (1+r)b_{t+1}.$$

Assume the following functional form:

$$u(c) = \frac{c^{1-\sigma} - 1}{1-\sigma}.$$

(a) Solve for the optimal values c_t^*, c_{t+1}^*, and b_{t+1}^*.

(b) What condition on y_t and y_{t+1} (taking other parameter values as given) is needed to make $b_{t+1}^* < 0$? Briefly discuss the intuition of this condition.

(c) Suppose that there is a large number of these agents, and they are all identical. They live in a 'closed economy', with no possibility of borrowing or lending from abroad. What does this imply about what consumption must be in equilibrium?

(d) Solve for the equilibrium interest rate that achieves this equilibrium condition. How does the ratio y_{t+1}/y_t affect the equilibrium r_t?

2. **Production and riskless bonds**. Suppose the household's problem is given by

$$\max_{c_t, b_{t+1}, k_{t+1}, c_{t+1}} \quad V_t = u\left(c_t\right) + \beta u\left(c_{t+1}\right)$$

subject to:

$$c_t = y_t - k_{t+1} - b_{t+1}$$
$$c_{t+1} = y_{t+1} + (1+r_t)b_{t+1}$$
$$y_{t+1} = F\left(k_{t+1}\right).$$

Note that this assumes complete depreciation of capital. Assume the following functional forms:

$$u(c) = \frac{c^{1-\sigma} - 1}{1-\sigma}$$
$$F(k_{t+1}) = \theta k_{t+1}^{1/2}.$$

(a) Solve for the optimal values $c_t^*, c_{t+1}^*, k_{t+1}^*$, and b_{t+1}^*.

(b) What conditions on y_t would make $b_{t+1}^* < 0$?

(c) Show (using comparative statics) how an increase in productivity (i.e. θ) affects c_t^*, c_{t+1}^*, k_{t+1}^*, and b_{t+1}^*. Discuss each result.

3. **More on Markov Chains.** Assume the random variable $\{X_t\}$ is described by a symmetric 2-state Markov chain $(x, \mathbb{P}, \pi_0)$, with state space $x' = [\mu - \sigma \quad \mu + \sigma]$, where μ is a constant and $\sigma > 0$, and with transition matrix given by

$$\mathbb{P} = \begin{bmatrix} p & 1-p \\ 1-p & p \end{bmatrix}$$

with $0 < p < 1$.

   (a) Compute, by hand, the stationary distribution of this Markov chain. Is it unique?

   (b) Use the previous result to show that μ and σ are, respectively, the mean and the standard deviation of the ergodic distribution of X_t.

4. **Constant Relative Risk Aversion Utility.**

   (a) Create a consumption vector with values for consumption from 0 to 10, in increments of 0.1. This will have 101 elements. Use this vector to create a 5×101 matrix, with the rows giving the CRRA utility values from each of the 101 consumption values, for 5 different values of σ: $\sigma = \{5, 3, 1.5, 1.1, 0.5\}$.

$$u(c) = \frac{c^{1-1/\sigma} - 1}{1 - 1/\sigma}.$$

   (b) Show that if $\sigma \to 1$ the utility function converges to $\ln(c)$.

   (c) Plot the five utility functions in a figure, with the consumption values on the horizontal axis.

   (d) Plot the utility function with $\sigma = 1.1$ along with $\ln(c)$. Did you expect this result?

   (e) Now repeat 4c–4d, but utilize the following utility function:

$$u(c) = \frac{c^{1-1/\sigma}}{1 - 1/\sigma}.$$

   Explain what you observe.

5. **Calculating Discounted Expected Utility.** Consider an infinitely lived consumer who has time-separable preferences, with single-period utility given by $u(c_t) = \frac{c_t^{1-1/\sigma} - 1}{1-1/\sigma}$, and a discount factor of $\beta = 0.95$. Consumption is *exogenously determined*—this is not a dynamic programming problem in which it is optimally chosen—and follows a 3-state Markov chain that takes on values in $\begin{bmatrix} 3 & 4 & 5 \end{bmatrix}$ and has transition matrix

$$\mathbb{P} = \begin{bmatrix} .7 & .3 & 0 \\ .3 & .4 & .3 \\ 0 & .3 & .7 \end{bmatrix}.$$

   (a) Write down the functional equation that characterizes the consumer's discounted expected utility from the exogenous consumption stream. What are the dimensions of the value function?

   (b) Given an elasticity of intertemporal substitution, σ, of 1/2, numerically solve (by inverting the appropriate matrix, etc.) the functional equation in Part 5a.

   (c) Solve the functional equation again, this time by iterating on the functional equation until convergence .

(d) Show how the consumer's discounted expected utility is related to σ by re-solving the functional equation (using the approach in Part 5c) for 3 different values of σ: $\sigma = \{0.5, 2, 4\}$. Explain the intuition of your findings.

(e) Repeat the last step, but this time leave the -1 term out of the numerator of the utility function. Can one compare the expected utility across different values of σ in a meaningful way? Explain.

6. **Numerical solution to a simple consumption-saving problem.** Consider an agent who maximizes

$$\sum_{t=1}^{T} \beta^{t-1} \frac{c_t^{1-1/\sigma} - 1}{1 - 1/\sigma}$$

subject to the budget constraint:

$$c_t + a_{t+1} \leq y_t + a_t(1 + r)$$

with initial assets a_0 given. Assets must remain positive in every period (i.e. $a_t > 0, \forall t$).

Use the supplied Matlab programs, `consumptionsaving.m` and `objective.m`, to answer/examine the following questions. Read through the code to understand its structure. You will need to slightly modify the code in order to adapt it to each question. The baseline parameter values are $T = 10$, $\sigma = 2$, $\beta = 1/(1 + 0.04)$, $r = 0.04$, $a_0 = 1$, and income equals 1 in every period.

(a) For the baseline parameters, solve the problem and plot the first-period value function and first-period policy functions (the decision rules for consumption and for next period's assets) and discuss their shapes.

(b) For the baseline parameters, use the decision rules to simulate consumption, wealth, and income, starting from initial wealth $a_0 = 1$. Explain the intuition behind the simulations.

(c) Now set $r = 0.02$, re-solve the value function, and again plot the simulations of consumption, wealth, and income, starting from initial wealth $a_0 = 1$. Explain the intuition of why things have changed, relative to the baseline.

(d) Leave $r = 0.02$, and increase the elasticity of intertemporal substitution by increasing σ to 10. Compare the simulations here to the simulations from the previous question.

(e) Set r back to its baseline value, set income in the first period equal to $\sum_{i=0}^{9} \left(\frac{1}{1+r}\right)^i \cdot 1$ and income in all other periods equal to zero. Create a new grid for assets that has an upper bound of 10, and then re-solve. Compare the simulations to the baseline simulations and explain the intuition of what you see.

(f) Now set income equal to 1 in periods 1–5, equal to 2 in periods 6–8, and again equal to 1 in periods 9–10. Re-solve the model and plot the simulations. Explain the intuition behind the simulations.

(g) Set $r = 0.02$ and let's allow $T \to \infty$. Let income equal 1 in every period. To solve this infinite-horizon problem, you must change the structure of the main loop so that rather than looping backward in time, the loop simply iterates until the value function converges (for some appropriate stopping criterion). Plot the value function.